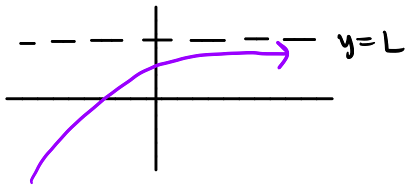
LECTURE: 2-6 LIMITS AT INFINITY

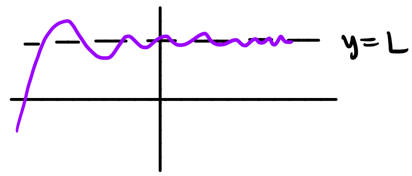
Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞) or $(-\infty, a)$. Then

$$\lim_{x\to\infty} f(x) = L \quad \left(\text{or } \lim_{x\to-\infty} f(x) = L\right)$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be big enough or

ex|



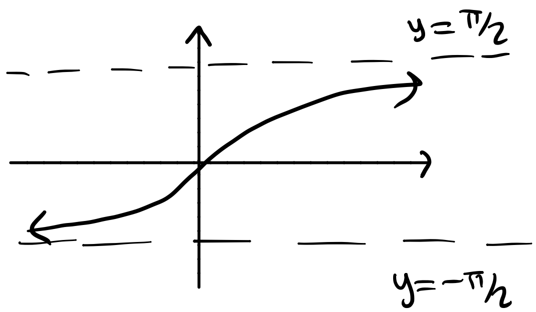


The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Example 1: Sketch a graph of $y = \tan^{-1} x$ and find the $\lim_{x\to\infty} \tan^{-1} x$ and $\lim_{x\to-\infty} \tan^{-1} x$.

Recall:

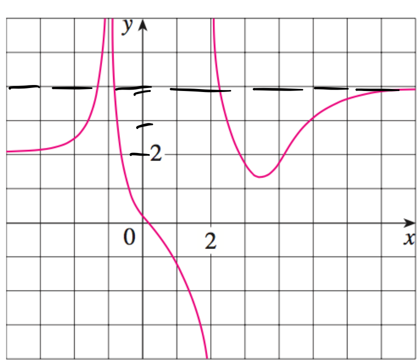


Thus, $\lim_{x\to\infty} \tan^{-1} x = \boxed{\pi/2}$

and $\lim_{x\to-\infty} \tan^{-1} x = \boxed{-\pi/2}$

sec. 2.2 sec. 2.6 or $x \to \pm\infty$

Example 2: Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown below.



infinite limits: $\lim_{x\to-1} f(x) = \infty$

$\lim_{x\to 2^-} f(x) = -\infty$

$\lim_{x\to 2^+} f(x) = \infty$

limits at infinity $\lim_{x\to\infty} f(x) = 4$

$\lim_{x\to-\infty} f(x) = 2$

asymptotes vertical: $\boxed{x=-1,\ x=2}$

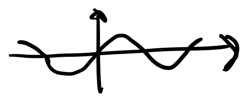
horizontal: $\boxed{y=2,\ y=4}$

Example 2: Find the following limits.

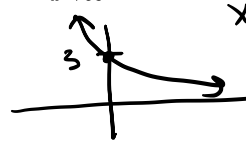
a) $\lim_{x\to\infty} \frac{1}{7x+1} = \boxed{0}$

$1 \div (\text{big \#}) \to 0$

b) $\lim_{x\to\infty} \sin x$ $\boxed{\text{DNE}}$



c) $\lim_{x\to\infty} 3e^{-x} = \lim_{x\to\infty} 3/e^x = 0$



How to Determine Limits at Infinity: Divide the numerator and denominator by the highest common power between the numerator and denominator.

Example 3: Find the limit.

(a) $\lim_{x\to\infty} \frac{(2x+5)\,1/x}{(x-4)\,1/x} = \lim_{x\to\infty} \frac{2+5/x}{1-4/x}$

$= \frac{2+0}{1-0}$

$= \boxed{2}$

(b) $\lim_{x\to\infty} \frac{(x+4)\,1/x}{(x^2+x-3)\,1/x} = \lim_{x\to\infty} \frac{1+4/x}{x+1-3/x}$

$= \boxed{0}$

$(1 \div (\text{big \#}) \longrightarrow 0)$

Example 4: Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{(2x^2+5)\,1/x^2}{(3x^2+1)\,1/x^2}$

$= \lim_{x\to\infty} \frac{2+5/x^2}{3+1/x^2}$

$= \frac{2+0}{3+0}$

$= \boxed{2/3}$

(b) $\lim_{x\to\infty} \frac{(2x+5)\,1/x}{(3x^2+1)\,1/x}$

$= \lim_{x\to\infty} \frac{2+5/x}{3x+1/x}$

$= \boxed{0}$

(c) $\lim_{x\to\infty} \frac{(2x^3+5)\,1/x^2}{(3x^2+1)\,1/x^2}$

$= \lim_{x\to\infty} \frac{2x+5/x^2}{3+1/x^2}$

$= \boxed{\infty}$

$(\text{big \#}) * 2 \div 3 \longrightarrow \infty$

Example 5: Find the following limits at infinity.

(a) $\lim\limits_{x\to\infty} \frac{(1+5e^x)\,1/e^x}{(7-e^x)\,1/e^x} = \lim\limits_{x\to\infty} \frac{(1/e^x + 5)}{(7/e^x - 1)}$

$= \boxed{-5}$

(b) $\lim\limits_{x\to\infty} [\ln(2+x) - \ln(1+x)] = \lim\limits_{x\to\infty} \ln\left(\frac{2+x}{1+x}\right)$

$= \ln\left(\lim\limits_{x\to\infty} \frac{(2+x)\,1/x}{(1+x)\,1/x}\right)$

$= \ln\left(\lim\limits_{x\to\infty} \frac{(2/x+1)}{(1/x+1)}\right)$

$= \ln(1)$

$= \boxed{0}$

Example 6: Find the limit.

(a) $\lim\limits_{x\to\infty} \frac{(x+2)\,1/x}{(\sqrt{9x^2+1})\,1/x} = \lim\limits_{x\to\infty} \frac{1+2/x}{\sqrt{(9x^2+1)\,1/x^2}}$

$= \lim\limits_{x\to\infty} \frac{1+2/x}{\sqrt{9+1/x^2}}$

$= \frac{1+0}{\sqrt{9+0}}$

$= \boxed{1/3}$

(b) $\lim\limits_{x\to\infty} \frac{(\sqrt{3x^6-x})\,1/x^3}{(x^3+1)\,1/x^3} = \lim\limits_{x\to\infty} \frac{\sqrt{(3x^6-x)\,1/x^6}}{1+1/x^3}$

$= \lim\limits_{x\to\infty} \frac{\sqrt{3-1/x^5}}{1+1/x^3}$

$= \frac{\sqrt{3-0}}{1+0}$

$= \boxed{\sqrt{3}}$

plugging in large, negatives. Think of this in two steps.

How do deal with limits as $x \to -\infty$: Replace x by $-x$ and take the limit as $x \to \infty$.

Example 7: Find the limit.

(a) $\lim\limits_{x\to-\infty} \frac{2x}{\sqrt{x^2+2}} = \lim\limits_{x\to\infty} \frac{2(-x)}{\sqrt{(-x)^2+2}}$

$= \lim\limits_{x\to\infty} \frac{(-2x)\cdot 1/x}{\sqrt{x^2+2}\cdot 1/x}$

$= \lim\limits_{x\to\infty} \frac{-2}{\sqrt{(x^2+2)\,1/x^2}}$

$= \lim\limits_{x\to\infty} \frac{-2}{\sqrt{1+2/x^2}}$

$= \frac{-2}{\sqrt{1+0}}$

$= \boxed{-2}$

(b) $\lim\limits_{x\to-\infty} (5-3e^x) = \lim\limits_{x\to\infty} (5-3e^{-x})$

$= \lim\limits_{x\to\infty} (5 - 3/e^x)$

$= 5 - 0$

$= \boxed{5}$

note, these are $\infty - \infty$... which is not zero...

Example 8: Evaluate the following limits.

(a) $\lim_{x\to\infty} (\sqrt{x^4+6x^2} - x^2)\left(\frac{\sqrt{x^4+6x^2}+x^2}{\sqrt{x^4+6x^2}+x^2}\right)$

$$= \lim_{x\to\infty} \frac{x^4+6x^2-x^4}{\sqrt{x^4+6x^2}+x^2}$$

$$= \lim_{x\to\infty} \frac{(6x^2)\,1/x^2}{(\sqrt{x^4+6x^2}+x^2)\,1/x^2}$$

$$= \lim_{x\to\infty} \frac{6}{\sqrt{(x^4+6x^2)1/x^4}+1}$$

$$= \lim_{x\to\infty} \frac{6}{\sqrt{1+6/x^2}+1}$$

$$= \frac{6}{1+1}$$

$$= \boxed{3}$$

(b) $\lim_{x\to\infty} (\sqrt{x^2+1} - x)\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$

$$= \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}$$

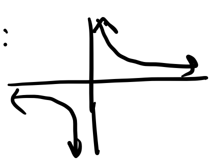
$$= \boxed{0}$$

Example 9: Evaluate the following limits.

(a) $\lim_{x\to 0^-} e^{1/x}$

as $x \to 0^-$, $1/x \to -\infty$.

see:



So $\lim_{x\to 0^-} e^{1/x} = \left(e^{-\text{big }\#}\right)$

$$= \boxed{0}$$

[Squeeze it!]

(b) $\lim_{x\to\infty} e^{-2x}\cos x = \lim_{x\to\infty} \frac{\cos x}{e^{2x}}$

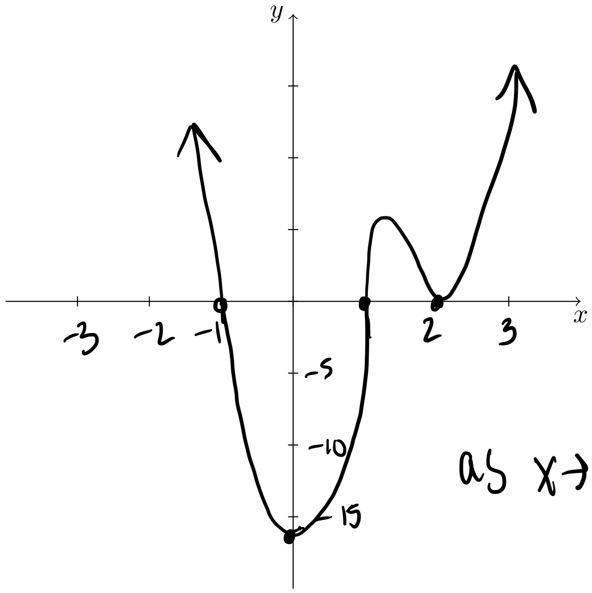
Note: $-1 \leq \cos x \leq 1$

and $-1/e^{2x} \leq \frac{\cos x}{e^{2x}} \leq 1/e^{2x}$

Since $\lim_{x\to\infty} 1/e^{2x} = 0$ and $\lim_{x\to\infty} -1/e^{2x} = 0$

we know $\lim_{x\to\infty} \frac{\cos x}{e^{2x}} = 0$ by the squeeze theorem.

Example 11: Sketch the graph of $y = (x-2)^4(x+1)^3(x-1)$ by finding its intercepts and its limits as $x \to \pm\infty$.



• y-intercept: $x=0 \Rightarrow y = (-2)^4(1)^3(-1)$

$y = -16$

• x-int: $y=0 \Rightarrow 0 = (x-2)^4(x+1)^3(x-1)$

$x = 2, -1, 1$

as $x \to \infty$, $y \to \infty$

as $x \to -\infty$, $y \to$ (big pos)(big neg)(big neg)

$y \to \infty$

Example 12: Find the horizontal and vertical asymptotes of $f(x) = \dfrac{\sqrt{16x^2+1}}{2x-8}$.

HA: $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\sqrt{16x^2+1} \cdot 1/x}{(2x-8)\,1/x}$ and $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{\sqrt{16x^2+1}}{2x-8}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{16+1/x^2}}{2-8/x}$

$= \dfrac{\sqrt{16+0}}{2-0}$

$\boxed{y=2}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{16(-x)^2+1}}{2(-x)-8}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{16x^2+1}\,(1/x)}{(-2x-8)\,1/x}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{16+1/x^2}}{-2-8/x}$

$= \dfrac{\sqrt{16+0}}{-2-0}$

$\boxed{y=-2}$

VA when $2x-8=0$

$2x=8$

$\boxed{x=4}$